Transportation Cost Inequalities for Neutral Functional SDEs

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Abstract

A class of functional differential equations are investigated. Using the Girsanov-transformation argument we establish the quadratic transportation cost inequalities for a class of finite-dimensional neutral functional stochastic differential equations and infinite-dimensional neutral functional stochastic partial differential equations under different metrics.

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1 Introduction

Let (E, d) be a metric space equipped with a σ-algebra $\mathscr{B}(E)$ such that metric $\mathrm{d}(\cdot,\cdot)$ is $\mathscr{B}(E)\times\mathscr{B}(E)$ measurable. For any $p \geq 1$ and probability measures μ and ν on $(E, \mathscr{B}(E), \mathrm{d})$, the L^p-Wasserstein distance between μ and ν is defined by

$$W_{p,\mathrm{d}}(\mu,\nu) := \inf_{\pi\in\mathscr{C}(\mu,\nu)} \Big\{ \int_{E\times E} \mathrm{d}^p(x,y)\pi(\mathrm{d}x,\mathrm{d}y) \Big\}^{1/p},$$

where $\mathscr{C}(\mu,\nu)$ denotes the totality of probability measures on $E\times E$ with the marginals μ and ν. In many practical situations, it is quite useful to find an upper bound for the metric $W_{p,\mathrm{d}}(\mu,\nu)$, where a fully satisfactory one given by Talagrand [16] is the relative entropy of ν with respect to μ

$$\mathrm{H}(\nu|\mu) := \begin{cases} \int \ln \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mathrm{d}\nu, & \nu \ll \mu, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\nu \ll \mu$ means that ν is absolutely continuous with respect to μ. We say that the probability measure μ on (E, d) satisfies an L^p transportation cost inequality (TCI) with a constant $C > 0$ if

$$W_{p,d}^2(\mu, \nu) \le 2C\mathrm{H}(\nu|\mu)$$

for any probability measure ν. As usual, we write $\mu \in T_p(C)$ for this relation.

Concentration inequalities and their applications have become an integral part of modern probability theory. Here we highlight three monographs: Ledoux [8], Üstünel [17] and Villani [19]. One of the powerful tools to show such concentration estimates for diffusions is TCI, where quadratic TCI is unique in its advantages and related to the log-Sobolev inequality, hypercontractivity, Poincaré inequality, inf-convolution, and Hamilton-Jacobi equations, for details, see, e.g., Bobkov and Götze [1], Gozlan, Roberto and Samson [6] Otto and Villani [12]. Recently, there are extensive literature on the topic of transportation cost inequalities (TCIs) in all kinds of path spaces, e.g., in [3] Djellout, Guilin and Wu investigate the stability under weak convergence, $T_1(C), T_2(C)$ and applications for random dynamical systems on Wiener space, in [14] Pal shows that probability laws of certain multidimensional semimartingales satisfy quadratic TCI, in [18] Üstünel proves TCI for the laws of diffusion processes where the drift depends on the full history. On Poisson space Wu [21] develops the W_1H inequality for stochastic differential equation (SDEs) with pure jumps, and Ma [10] discusses the W_1H inequality for SDEs driven by the Brownian motion and the jump together. Wang [20] establishes some TCIs on the path space over a connected complete Riemannian manifold with Ricci curvature bounded from below. Fang and Shao [4] provides optimal transport maps for Monge-Kantorovich problem on connected Lie groups, to name a few.

Moreover, many other methods have been applied to establish TCIs, e.g., Large deviation principle is used in [5], and ensorization is implemented in [20]. In particular, the Girsanov transformation theorem has been utilized in [3] to provide a characterization of L^1-TCIs for diffusions, which has been generalized to different setting, e.g., infinite-dimensional dynamical systems in [23], time-inhomogenous diffusions in [14], multivalued SDEs and singular SDEs in [18], and SDEs driven by a fractional Brownian motion in [15]. On the other hand, due to the Girsanov transformation is unavailable for the jump cases, recently Malliavin calculus technique is applied in [10, 21].

Motivated by the previous works, and references therein, we shall establish the T_2-TCIs for a class of finite-dimensional neutral functional SDEs and infinite-dimensional neutral functional stochastic partial differential equations (SPDEs). A neutral functional differential equation (see [7, 9, 11]) is one in which the derivatives of the past history are involved, as well as those of the present state of the system.

The rest of the paper is organised as follows: the T_2-TCIs for a class of finite-dimensional neutral functional SDEs are studied in Section 2, in Section 3 we show the T_2-TCIs for infinite-dimensional neutral functional stochastic partial differential equations (SPDEs).

2 TCIs for Neutral Functional SDEs

For a positive integer n, let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$ be the Euclidean space and $\|A\|_{HS} := \sqrt{\mathrm{trace}(A^*A)}$, the Hilbert-Schmidt norm for a matrix A, where A^* is the transpose of A. Let $\tau > 0$ and $T > 0$ be two constants. Denote $\mathscr{C} := \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ by the family of continuous functions $\xi : [-\tau, 0] \mapsto \mathbb{R}^n$ and $\mathcal{C}([-\tau, T]; \mathbb{R}^n)$ the set of continuous functions on $[-\tau, T]$. Let $X \in \mathcal{C}([-\tau, T]; \mathbb{R}^n)$ and $t \in [0, T]$, we define the segment $X_t \in \mathscr{C}$ of X by $X_t(\theta) := X(t+\theta)$ for $\theta \in [-\tau, 0]$, where $X(t) \in \mathbb{R}^n$ is a point, while $X_t \in \mathscr{C}$ is a continuous $\mathbb{R}^n$-valued function on $[-\tau, 0]$. Denote $\mathcal{W}([-\tau, 0]; \mathbb{R}_+)$ by the family of Borel-measurable functions $w : [-\tau, 0] \mapsto \mathbb{R}_+$ such that $\int_{-\tau}^0 w(\theta)\mathrm{d}\theta = 1$. Throughout the paper $C > 0$ is a generic constant whose values may change for its different appearances.

Consider neutral functional SDE on $\mathbb{R}^n$

eq1
$$\begin{cases} \mathrm{d}[X(t) - G(X_t)] = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t), \quad t \in [0, T], \\ X_0 = \xi \in \mathscr{C}. \end{cases} \tag{2.1}$$

Here $G, b : \mathscr{C} \to \mathbb{R}^n$, $\sigma : \mathscr{C} \to \mathbb{R}^{n\times m}$, and W is an m-dimensional Brownian motion defined on some filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Assume that $G(0) = 0$ and there exist $\kappa \in (0, 1)$ and $w_1 \in \mathcal{W}([-\tau, 0]; \mathbb{R}_+)$ such that

eq7
$$|G(\xi) - G(\eta)|^2 \leq \kappa \int_{-\tau}^0 w_1(\theta)|\xi(\theta) - \eta(\theta)|^2\mathrm{d}\theta, \quad \xi, \eta \in \mathscr{C}. \tag{2.2}$$

We further assume that b and σ are locally Lipschitzian and there exist $\delta > 0, \lambda_1 \in \mathbb{R}, \lambda_2 \geq 0$ and $w_2 \in \mathcal{W}([-\tau, 0]; \mathbb{R}_+)$ such that

eq9
$$2\langle\xi(0) - G(\xi), b(\xi)\rangle + \|\sigma(\xi)\|_{HS}^2 \leq \delta\Big(1 + |\xi(0)|^2 + \int_{-\tau}^0 |\xi(\theta)|^2\mathrm{d}\theta\Big), \quad \xi \in \mathscr{C}, \tag{2.3}$$

and for arbitrary $\xi, \eta \in \mathscr{C}$

eq8
$$\begin{aligned} &2\langle\xi(0) - G(\xi) - (\eta(0) - G(\eta)), b(\xi) - b(\eta)\rangle + \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \\ &\qquad \leq -\lambda_1|\xi(0) - \eta(0)|^2 + \lambda_2 \int_{-\tau}^0 w_2(\theta)|\xi(\theta) - \eta(\theta)|^2\mathrm{d}\theta. \end{aligned} \tag{2.4}$$

Remark 2.1. If we take $G(\xi) = G(\xi(-\tau)), b(\xi) = b(\xi(0), \xi(-\tau)), \sigma(\xi) = \sigma(\xi(0), \xi(-\tau)), \xi \in \mathscr{C}$, then the equation (2.1) becomes a delay SDE, that is:

$$\begin{cases} \mathrm{d}[X(t) - G(X(t-\tau)] = b(X(t), X(t-\tau))\mathrm{d}t + \sigma(X(t), X(t-\tau))\mathrm{d}W(t), \quad t \in [0, T], \\ X_0 = \xi \in \mathscr{C}. \end{cases}$$

There are many examples satisfying (2.2)-(2.4). Actually (2.2) becomes $|G(\xi(-\tau)) - G(\eta(-\tau))| \leq \kappa|\xi(-\tau) - \eta(-\tau)|$ for some $\kappa \in (0, 1)$ by taking $w(\theta) = 1/\tau$ for $\theta \in [-\tau, 0]$ (In fact, for the delay we can take $\kappa \geq 1$ by an induction argument). Moreover, since b and σ are locally Lipschitzian, Eq. (2.1) has a unique local solution. On the other hand, (2.3) can suppress the explosion of the solution in a finite-time interval. Therefore Eq. (2.1) has a unique solution $\{X(t, \xi)\}$ on $[0, T]$.

Let $T = N$ be a positive integer. For any $\gamma_1, \gamma_2 \in \mathcal{X} := \mathcal{C}([0,T];\mathbb{R}^n)$, we define

$$\mathrm{d}_{\infty,1}(\gamma_1,\gamma_2) := \Big\{\sum_{k=0}^{N-1} \sup_{k\le t\le k+1} |\gamma_1(t)-\gamma_2(t)|^2\Big\}^{1/2}, \quad \gamma_1,\gamma_2 \in \mathcal{X}.$$

Then $\mathcal{X}$ is a Banach space with respect to $\mathrm{d}_{\infty,1}$.

Theorem 1 **Theorem 2.1.** Let (2.2)-(2.4) hold and $\mathbb{P}_\xi$ be the law of $X(\cdot,\xi)$, solution process of Eq. (2.1). Assume that σ is bounded by $\tilde\sigma := \sup_{\xi\in\mathscr{C}} \|\sigma(\xi)\|_{HS} < \infty$, $\lambda_1 - \lambda_2 > 0$, and there exists $\lambda_3 > 0$ such that

eq38
$$\|\sigma(\xi)-\sigma(\eta)\|_{HS}^2 \le \lambda_3 \int_{-\tau}^0 |\xi(\theta)-\eta(\theta)|^2 \mathrm{d}\theta, \quad \xi,\eta\in\mathscr{C}. \tag{2.5}$$

Then $\mathbb{P}_\xi \in T_2(C)$ on $\mathcal{X}$ under the metric $\mathrm{d}_{\infty,1}$, where $C > 0$ is dependent on $\kappa, \lambda_i, i = 1,2,3, \tilde\sigma, N$ and the universal constant from the Burkhold-Davis-Gundy inequality. Moreover, under the previous assumptions, except (2.5), $\mathbb{P}_\xi \in T_2(C)$ for some $C > 0$ under the metric

$$\mathrm{d}_{L^2}(\gamma_1,\gamma_2) := \Big(\int_0^T |\gamma_1(t)-\gamma_1(t)|^2\mathrm{d}t\Big)^{\frac12}, \quad \gamma_1,\gamma_2\in\mathcal{X},$$

where $C > 0$ is independent of T whenever $\lambda_1 - \lambda_2 > 0$, otherwise dependent on T.

Proof. It should be pointed out that the argument is motivated by that of [3, 22], where the key point is to express the finiteness of the entropy by means of the energy of the drift from the Girsanov transformation of a well chosen probability measure. In the sequel we divide the proof into two steps to show the desired assertions under two different metrics.

(**1**) Let $\mathbb{P}_\xi$ be the law of $X(\cdot,\xi)$ on $\mathcal{X}$ and $\mathbb{Q}$ be any probability measure on $\mathcal{X}$ such that $\mathbb{Q} \ll \mathbb{P}_\xi$. Define

eq17
$$\tilde{\mathbb{Q}} := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_\xi}(X(\cdot,\xi))\mathbb{P}, \tag{2.6}$$

which is a probability measure on $(\Omega,\mathcal{F})$. Recalling the definition of "entropy" and adopting a measure-transformation argument we obtain from (2.6) that

$$\begin{aligned}\mathbf{H}(\tilde{\mathbb{Q}}|\mathbb{P}) &= \int_\Omega \ln\Big(\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big)\mathrm{d}\tilde{\mathbb{Q}} = \int_\Omega \ln\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_\xi}(X(\cdot,\xi))\Big)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_\xi}(X(\cdot,\xi))\mathrm{d}\mathbb{P}\\ &= \int_{\mathcal{X}} \ln\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_\xi}\Big)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_\xi}\mathrm{d}\mathbb{P}_\xi\\ &= \mathbf{H}(\mathbb{Q}|\mathbb{P}_\xi).\end{aligned}$$

Additionally, by the martingale representation theorem, there exists by, e.g., [3, 22], a predictable process $h \in \mathbb{R}^m$ with $\int_0^t |h(s)|^2\mathrm{d}s < \infty$ for $t\in[0,N]$, $\mathbb{P}$-a.s., such that

eq4
$$\mathbf{H}(\tilde{\mathbb{Q}}|\mathbb{P}) = \mathbf{H}(\mathbb{Q}|\mathbb{P}_\xi) = \frac12\mathbb{E}^{\tilde{\mathbb{Q}}}\int_0^N |h(t)|^2\mathrm{d}t. \tag{2.7}$$

Furthermore, due to the Girsanov theorem,

$$\tilde{W}(t) := W(t) - \int_0^t h(s)\mathrm{d}s, \quad t \in [0, N],$$

is a Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{Q}})$. Then, under the measure $\tilde{\mathbb{Q}}$, the process $\{X(t, \xi)\}_{t\in[0,N]}$ satisfies

eq2
$$\begin{cases} \mathrm{d}[X(t) - G(X_t)] = [b(X_t) + \sigma(X_t)h(t)]\mathrm{d}t + \sigma(X_t)\mathrm{d}\tilde{W}(t), \quad t \in [0, N], \\ X_0 = \xi. \end{cases} \tag{2.8}$$

Observe that, to show $\mathbb{P}_\xi \in T_2(C)$ for some $C > 0$, we need to bound the Wasserstein distance between $\mathbb{P}_\xi$ and $\mathbb{Q}$, and couple the solutions of Eq. (2.1) and Eq. (2.8). Let $\{Y(t, \xi)\}_{t\in[0,N]}$ be the solution of the following equation

eq3
$$\begin{cases} \mathrm{d}[Y(t) - G(Y_t)] = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}\tilde{W}(t), \quad t \in [0, N], \\ Y_0 = \xi. \end{cases} \tag{2.9}$$

By virtue of the uniqueness, under $\tilde{\mathbb{Q}}$ the law of $Y(\cdot, \xi)$ is $\mathbb{P}_\xi$. Consequently (X, Y) under $\tilde{\mathbb{Q}}$ is a coupling of $(\mathbb{Q}, \mathbb{P}_\xi)$ and, by the definition of "L^2-Wasserstein distance", we have

eq18
$$(W_{2,\mathrm{d}}(\mathbb{Q}, \mathbb{P}_\xi))^2 \leq \mathbb{E}^{\tilde{\mathbb{Q}}}(\mathrm{d}^2_{\infty,1}(X, Y)). \tag{2.10}$$

Thus, by (2.7) and (2.10), to obtain the desired Talagrand's inequality, it suffices to estimate the $\mathrm{d}_{\infty,1}$-distance between X and Y, which is bounded by $\mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^N |h(t)|^2\mathrm{d}t$ up to some constant. That is, we only need to verify that for $t \in [0, N]$

eq13
$$\mathbb{E}^{\tilde{\mathbb{Q}}}(\mathrm{d}^2_{\infty,1}(X, Y)) = \sum_{k=0}^{N-1} \mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\leq t\leq k+1} |X(t) - Y(t)|^2\Big) \leq C\mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^N |h(t)|^2\mathrm{d}t. \tag{2.11}$$

Recall a fundamental inequality: for any $a, b > 0$ and $\epsilon \in (0, 1)$,

eq6
$$(a + b)^2 \leq a^2/\epsilon + b^2/(1 - \epsilon). \tag{2.12}$$

This, together with (2.2), yields that

$$\mathbb{E}^{\tilde{\mathbb{Q}}}|X(t) - Y(t)|^2 \leq \frac{1}{1 - \sqrt{\kappa}}\mathbb{E}^{\tilde{\mathbb{Q}}}|M(t)|^2 + \sqrt{\kappa} \int_{-\tau}^0 w_1(\theta)\mathbb{E}^{\tilde{\mathbb{Q}}}|X(t + \theta) - Y(t + \theta)|^2\mathrm{d}\theta,$$

where $M(t) := X(t) - G(X_t) - (Y(t) - G(Y_t)), t \in [0, N]$. Due to $w_1 \in \mathcal{W}([-\tau, 0]; \mathbb{R}_+)$ and $X(\theta) = Y(\theta)$ for $\theta \in [-\tau, 0]$, it is easy to note that

eq12
$$\sup_{0\leq s\leq t} \mathbb{E}^{\tilde{\mathbb{Q}}}|X(s) - Y(s)|^2 \leq \frac{1}{(1 - \sqrt{\kappa})^2} \sup_{0\leq s\leq t} \mathbb{E}^{\tilde{\mathbb{Q}}}|M(s)|^2, \quad t \in [0, N]. \tag{2.13}$$

Furthermore, by the Itô formula and the Young inequality, together with (2.2), (2.4), the boundedness of σ and $w_1, w_2 \in \mathcal{W}([-\tau, 0]; \mathbb{R}_+)$, we obtain that for arbitrary $\epsilon \in (0,1)$

eq34

$$\begin{aligned}
&\mathbb{E}^{\tilde{\mathbb{Q}}}|M(t)|^2 \\
&\le -\lambda_1 \mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t |X(s) - Y(s)|^2 \mathrm{d}s + \lambda_2 \mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t \int_{-\tau}^0 w_2(\theta)|X(s+\theta) - Y(s+\theta)|^2 \mathrm{d}\theta \mathrm{d}s \\
&\quad + 2\tilde{\sigma}\mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t |M(s)||h(s)|\mathrm{d}s \\
&\le -\lambda_1 \mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t |X(s) - Y(s)|^2 \mathrm{d}s + \lambda_2 \mathbb{E}^{\tilde{\mathbb{Q}}} \int_{-\tau}^0 w_2(\theta) \int_0^t |X(s) - Y(s)|^2 \mathrm{d}s\mathrm{d}\theta \\
&\quad + \mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t \Big\{ \frac{\epsilon}{1+\kappa}\Big(|X(s) - Y(s)|^2 + \kappa \int_{-\tau}^0 w_1(\theta)|X(s+\theta) - Y(s+\theta)|^2 \mathrm{d}\theta\Big) \\
&\quad + \frac{2(1+\kappa)\tilde{\sigma}^2}{\epsilon}|h(s)|^2\Big\}\mathrm{d}s \\
&\le -\tilde{\lambda}\mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t |X(s) - Y(s)|^2 \mathrm{d}s + \frac{2(1+\kappa)\tilde{\sigma}^2}{\epsilon} \int_0^t \mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2 \mathrm{d}s, \quad t \in [0, N],
\end{aligned} \tag{2.14}$$

where $\tilde{\lambda} := \lambda_1 - \lambda_2 - \epsilon$, and we have also used the fact that the stochastic integral has zero expectation under $\tilde{\mathbb{Q}}$ by a standard stopping time trick. Choose $\epsilon \in (0,1)$ sufficiently small such that $\tilde{\lambda} > 0$ and note that $t \in [0, N]$,

eq45

$$\sup_{0\le s\le t} \mathbb{E}^{\tilde{\mathbb{Q}}}|M(s)| + \tilde{\lambda}\mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t |X(s) - Y(s)|^2 \mathrm{d}s \le 2 \sup_{0\le s\le t} \mathbb{E}^{\tilde{\mathbb{Q}}}\Big\{|M(s)| + \int_0^s |X(r) - Y(r)|^2 \mathrm{d}r\Big\} \tag{2.15}$$

By (2.13) and (2.14) we have

$$\mathbb{E}^{\tilde{\mathbb{Q}}}|X(t) - Y(t)|^2 \le -\beta_1 \mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^t |X(s) - Y(s)|^2 \mathrm{d}s + \beta_2 \int_0^t \mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2 \mathrm{d}s, \quad t \in [0, N],$$

where $\beta_1 := \tilde{\lambda}/(1-\sqrt{\kappa})^2 > 0$ and $\beta_2 := 4(1+\kappa)\tilde{\sigma}^2/(\epsilon(1-\sqrt{\kappa})^2)$. This, along with the Gronwall inequality, yields that

eq35

$$\mathbb{E}^{\tilde{\mathbb{Q}}}|X(t) - Y(t)|^2 \le \beta_2 \int_0^t e^{-\beta_1(t-s)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2 \mathrm{d}s, \quad t \in [0, N]. \tag{2.16}$$

Also by the Itô formula and the Young inequality we can deduce from (2.2) and (2.4) that

for any $w_2 \in \mathcal{W}([-\tau,0];\mathbb{R}_+)$, $\epsilon>0$ and $t\in[k,k+1), k=0,1,\cdots,N-1$,

$$
\begin{aligned}
|M(t)|^2 &\le |M(k)|^2 - \lambda_1\int_k^t |X(s)-Y(s)|^2\mathrm{d}s + \lambda_2\int_{-\tau}^0 w_2(\theta)\int_k^t |X(s+\theta)-Y(s+\theta)|^2\mathrm{d}s\mathrm{d}\theta\\
&\quad + 2\tilde{\sigma}\int_k^t |M(s)||h(s)|\mathrm{d}s + 2\int_k^t \langle M(s), \sigma(X_s)-\sigma(Y_s)\mathrm{d}\tilde{W}(s)\rangle\\
&\le |M(k)|^2 + (\lambda_2+2\kappa\epsilon)\int_{k-\tau}^k |X(s)-Y(s)|^2\mathrm{d}s + C_1\int_k^t |X(s)-Y(s)|^2\mathrm{d}s\\
&\quad + \frac{\tilde{\sigma}^2}{\epsilon}\int_k^t |h(s)|^2\mathrm{d}s + 2\int_k^t \langle M(s), \sigma(X_s)-\sigma(Y_s)\mathrm{d}\tilde{W}(s)\rangle,
\end{aligned}
$$

where $C_1 := -\lambda_1+\lambda_2+2(1+\kappa)\epsilon$. Choosing $\epsilon>0$ such that $C_1>0$ we have

eq36

$$
\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|M(t)|^2\Big) &\le \mathbb{E}^{\tilde{\mathbb{Q}}}|M(k)|^2 + (\lambda_2+2\kappa\epsilon)\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^k |X(s)-Y(s)|^2\mathrm{d}s\\
&\quad + C_1\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}|X(s)-Y(s)|^2\mathrm{d}s\\
&\quad + \frac{\tilde{\sigma}^2}{\epsilon}\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}|h(s)|^2\mathrm{d}s + J(t),
\end{aligned}
\tag{2.17}
$$

where

$$
J(t) := 2\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}\Big|\int_k^t \langle M(s), \sigma(X_s)-\sigma(Y_s)\mathrm{d}\tilde{W}(s)\rangle\Big|\Big).
$$

By the Burkhold-Davis-Gundy inequality and the Young inequality, together with (2.5), it then follows that

eq42

$$
\begin{aligned}
J(t) &\le 6\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\int_k^{k+1}|M(s)|^2\|\sigma(X_s)-\sigma(Y_s)\|_{HS}^2\mathrm{d}s\Big)^{1/2}\\
&\le \frac{1}{2}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|M(t)|^2\Big) + 9\lambda_3\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}\int_{-\tau}^0 |X(s+\theta)-Y(s+\theta)|^2\mathrm{d}\theta\mathrm{d}s\\
&\le \frac{1}{2}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|M(t)|^2\Big) + 9\lambda_3\tau\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^{k+1}|X(s)-Y(s)|^2\mathrm{d}s.
\end{aligned}
\tag{2.18}
$$

Thus from (2.17) one has

eq37

$$
\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|M(t)|^2\Big) &\le 2\mathbb{E}^{\tilde{\mathbb{Q}}}|M(k)|^2 + 2(\lambda_2+2\kappa\epsilon)\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^k |X(s)-Y(s)|^2\mathrm{d}s\\
&\quad + 2C_1\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}|X(s)-Y(s)|^2\mathrm{d}s + \frac{\widetilde{2\sigma}^2}{\epsilon}\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}|h(s)|^2\mathrm{d}s\\
&\quad + 18\lambda_3\tau\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^{k+1}|X(s)-Y(s)|^2\mathrm{d}s.
\end{aligned}
\tag{2.19}
$$

Moreover note from (2.12) that

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|X(t)-Y(t)|^2\Big) &\le \frac{1}{(1-\sqrt{\kappa})^2}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|M(t)|^2\Big)\\ &\quad+\frac{\sqrt{\kappa}}{1-\sqrt{\kappa}}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k-\tau\le t\le k}|X(t)-Y(t)|^2\Big).\end{aligned}$$

On the other hand, by (2.2)

$$\mathbb{E}^{\tilde{\mathbb{Q}}}|M(k)|^2 \le 2\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k)-Y(k)|^2+2\int_{-\tau}^0 w_1(\theta)\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k+\theta)-Y(k+\theta)|^2\mathrm{d}\theta$$

As a result, due to (2.19) we obtain that

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|X(t)-Y(t)|^2\Big) &\le C\Big\{\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k-\tau\le t\le k}|X(t)-Y(t)|^2\Big)+\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}|h(s)|^2\mathrm{d}s\\ &\quad+\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k)-Y(k)|^2+\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^k|X(s)-Y(s)|^2\mathrm{d}s\\ &\quad+\int_{-\tau}^0 w_1(\theta)\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k+\theta)-Y(k+\theta)|^2\mathrm{d}\theta\Big\}\\ &\quad+C\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}|X(s)-Y(s)|^2\mathrm{d}s.\end{aligned}$$

An application of the Gronwall inequality further implies that

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|X(t)-Y(t)|^2\Big) &\le C\Big\{\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k-\tau\le t\le k}|X(t)-Y(t)|^2\Big)+\mathbb{E}^{\tilde{\mathbb{Q}}}\int_k^{k+1}|h(s)|^2\mathrm{d}s\\ &\quad+\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k)-Y(k)|^2+\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^k|X(s)-Y(s)|^2\mathrm{d}s\\ &\quad+\int_{-\tau}^0 w_1(\theta)\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k+\theta)-Y(k+\theta)|^2\mathrm{d}\theta\Big\}.\end{aligned}$$

Thus for any $1\le M\le N$ one has

$$\begin{aligned}\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|X(t)-Y(t)|^2\Big) &\le C\Big\{\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k-\tau\le t\le k}|X(t)-Y(t)|^2\Big)+\mathbb{E}^{\tilde{\mathbb{Q}}}\int_0^M|h(s)|^2\mathrm{d}s\\ &\quad+\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k)-Y(k)|^2+\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^k|X(s)-Y(s)|^2\mathrm{d}s\\ &\quad+\int_{-\tau}^0 w_1(\theta)\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k+\theta)-Y(k+\theta)|^2\mathrm{d}\theta\Big\}.\end{aligned}$$

Note from (2.16) that

$$\begin{aligned}\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}\int_{k-\tau}^{k}|X(s)-Y(s)|^2\mathrm{d}s &\le \beta_2\sum_{k=0}^{M-1}\int_{k-\tau}^{k}\int_0^s e^{-\beta_1(s-r)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(r)|^2\mathrm{d}r\mathrm{d}s\\ &=\beta_2\sum_{k=0}^{M-1}\int_{k-\tau}^{k}\int_0^s e^{-\beta_1(k-r)}e^{-\beta_1(s-k)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(r)|^2\mathrm{d}r\mathrm{d}s\\ &\le \beta_2 e^{\beta_1\tau}\tau\sum_{k=0}^{M-1}\int_0^k e^{-\beta_1(k-s)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2\mathrm{d}s,\end{aligned}$$

and due to $w_1\in\mathcal{W}([-\tau,0];\mathbb{R}_+)$

$$\begin{aligned}&\int_{-\tau}^0 w_1(\theta)\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}|X(k+\theta)-Y(k+\theta)|^2\mathrm{d}\theta\\ &\le\beta_2\int_{-\tau}^0 w_1(\theta)\sum_{k=0}^{M-1}\int_0^{k+\theta}e^{-\beta_1(k+\theta-s)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2\mathrm{d}s\mathrm{d}\theta\\ &\le\beta_2 e^{\beta_1\tau}\sum_{k=0}^{M-1}\int_0^k e^{-\beta_1(k-s)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2\mathrm{d}s.\end{aligned}$$

Then, carrying out a similar procedure to that of [22, Inquality (3.6), p363], we can derive that for any $1\le M\le N$

$$\sum_{k=0}^{M-1}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|X(t)-Y(t)|^2\Big)\le C\Big\{\sum_{k=0}^{M-2}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{k\le t\le k+1}|X(t)-Y(t)|^2\Big)+\mathbb{E}^{\tilde{\mathbb{Q}}}\int_0^N|h(s)|^2\mathrm{d}s\Big\},$$

and the claim (2.11) follows by a deduction argument.

(**2**) Observing the proceeding proof we can also deduce that

$$\mathbb{E}^{\tilde{\mathbb{Q}}}|X(t)-Y(t)|^2\le C\int_0^t e^{C(t-s)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2\mathrm{d}s,\quad t\in[0,T].$$

By changing the integral order it follows that

$$\begin{aligned}\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|X(t)-Y(t)|^2\mathrm{d}t&\le C\int_0^T\int_0^t e^{C(t-s)}\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2\mathrm{d}s\mathrm{d}t\\ &=C\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2\int_s^T e^{c_3(t-s)}\mathrm{d}t\mathrm{d}s\\ &\le C\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|h(s)|^2\mathrm{d}s,\end{aligned}$$

and then $\mathbb{P}_\xi\in T_2(C)$ for some $C>0$ under the metric d_{L^2}. The proof is therefore complete. $\square$

Remark 2.2. The processes $X(t)$ we consider throughout the paper is not a Markov process since G, b and σ are dependent on the past history of the solution process.

Observing the argument of Theorem 2.1 we can also obtain the following result.

Theorem 4 **Theorem 2.2.** Let the conditions in Theorem 2.1 hold. Then $\mathbb{P}_\xi \in T_2(C)$ for some constant $C > 0$, which is dependent only on $\kappa, \lambda_i, i = 1, 2, 3, \tilde{\sigma}$ and the universal constant from the Burkhold-Davis-Gundy inequality, under uniform metric

$$\mathrm{d}_{\infty,2}(\gamma_1, \gamma_2) := \sup_{0 \le t \le T} |\gamma_1(t) - \gamma_2(t)|, \quad \gamma_1, \gamma_2 \in \mathcal{X}.$$

In what follows we shall show $\mathbb{P}_\xi \in T_2(C)$ for some $C > 0$ with respect to the uniform metric $\mathrm{d}_{\infty,2}$ under weaker conditions than (2.2)-(2.4). For $\xi \in \mathscr{C}$, denote $\|\xi\|_\infty = \sup_{-\tau \le \theta \le 0} |\xi(\theta)|$. Assume that $G(0) = 0$ and there exists $\kappa \in (0, 1)$ such that

eq14
$$|G(\xi) - G(\eta)| \le \kappa \|\xi - \eta\|_\infty, \quad \xi, \eta \in \mathscr{C}. \tag{2.20}$$

Assume also that b and σ are locally Lipschizian and there exists $\tilde{\lambda}_1, \tilde{\lambda}_2 \ge 0$ such that for arbitrary $\xi, \eta \in \mathscr{C}$

eq15
$$2\langle \xi(0) - G(\xi), b(\xi) \rangle + \|\sigma(\xi)\|_{HS}^2 \le \tilde{\lambda}_1 (1 + \|\xi\|_\infty^2) \tag{2.21}$$

and

eq16
$$2\langle \xi(0) - G(\xi) - (\eta(0) - G(\eta)), b(\xi) - b(\eta) \rangle + \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \le \tilde{\lambda}_2 \|\xi - \eta\|_\infty^2. \tag{2.22}$$

Theorem 2 **Theorem 2.3.** Let (2.20)-(2.22) hold. Assume further that there exists $\tilde{\lambda}_3 \ge 0$ such that

eq39
$$\|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \le \tilde{\lambda}_3 \|\xi(\theta) - \eta(\theta)\|_\infty^2, \quad \xi, \eta \in \mathscr{C}, \tag{2.23}$$

and σ is bounded by $\tilde{\sigma} := \sup_{\xi \in \mathscr{C}} \|\sigma(\xi)\|_{HS} < \infty$. Then $\mathbb{P}_\xi \in T_2(C)$ for some $C > 0$ on the metric space $\mathcal{X}$ with respect to the metric $\mathrm{d}_{\infty,2}$.

Proof. Note that, under (2.20)-(2.22), Eq. (2.1) has a unique solution $\{X(t, \xi)\}_{t \in [-\tau, T]}$. We also point out that the argument is similar to that of Theorem 2.1, while a sketch of the proof is provided for the completeness and highlight some differences. Let $\tilde{\mathbb{Q}}$ be defined by (2.6). Adopting a similar procedure to that of (2.10), we have

eq5
$$(W_{2,\mathrm{d}}(\mathbb{Q}, \mathbb{P}_\xi))^2 \le \mathbb{E}^{\tilde{\mathbb{Q}}}(\mathrm{d}_{\infty,2}^2(X, Y)). \tag{2.24}$$

Thus by (2.7) and (2.24) to verify that $\mathbb{P}_\xi \in T_2(C)$ for some $C > 0$ under the metric $\mathrm{d}_{\infty,2}$ it is sufficient to verify that

eq11
$$\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0 \le t \le T} |X(t) - Y(t)|^2 \Big) \le C \mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^T |h(t)|^2 \mathrm{d}t. \tag{2.25}$$

Let $M(t) := X(t) - G(X_t) - (Y(t) - G(Y_t)), t \in [0, T]$. Recalling the inequality (2.12) and noting that $X_0 = Y_0 = \xi$, we then obtain from (2.20) that

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|X(t)-Y(t)|^2\Big) &\le \frac{1}{1-\kappa}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|M(t)|^2\Big)+\frac{1}{\kappa}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|G(X_t)-G(Y_t)|^2\Big)\\ &\le \frac{1}{1-\kappa}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|M(t)|^2\Big)+\kappa\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|X(t)-Y(t)|^2\Big).\end{aligned}$$

This implies that

eq10 (2.26)
$$\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|X(t)-Y(t)|^2\Big) \le \frac{1}{(1-\kappa)^2}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|M(t)|^2\Big).$$

Then, applying the Itô formula and the Burkhold-Davis-Gundy inequality, we can derive from (2.20), (2.22)and (2.23) that

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|M(t)|^2\Big) &\le (\tilde{\lambda}_2+2(1+\kappa^2))\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\Big)\mathrm{d}s+\tilde{\sigma}^2\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|h(t)|^2\mathrm{d}t\\ &\quad+6\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|M(t)|^2\int_0^T\|\sigma(X_s)-\sigma(Y_s)\|_{HS}^2\mathrm{d}s\Big)^{\frac{1}{2}}\\ &\le \Big(\tilde{\lambda}_2+2(1+\kappa^2)+18\tilde{\lambda}_3^2\Big)\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\Big)\mathrm{d}s\\ &\quad+\tilde{\sigma}^2\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|h(t)|^2\mathrm{d}t+\frac{1}{2}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|M(t)|^2\Big),\end{aligned}$$

where we have also used the boundedness of σ and the Young inequality in the last step. It then follows that

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|M(t)|^2\Big)\le C\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\Big)\mathrm{d}s+2\tilde{\sigma}^2\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|h(t)|^2\mathrm{d}t.$$

Thus from (2.26) we get

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|X(t)-Y(t)|^2\Big)&\le\frac{C}{(1-\kappa)^2}\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le s\le t}|X(s)-Y(s)|^2\Big)\mathrm{d}s\\ &\quad+\frac{2\tilde{\sigma}^2}{(1-\kappa)^2}\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|h(t)|^2\mathrm{d}t,\end{aligned}$$

and due to the Gronwall inequality

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\le t\le T}|X(t)-Y(t)|^2\Big)\le C\int_0^T\mathbb{E}^{\tilde{\mathbb{Q}}}|h(t)|^2\mathrm{d}t.$$

Consequently the statement (2.25) follows and thus $\mathbb{P}_\xi \in T_2(C)$ holds for some $C > 0$, as required. $\square$

Remark 2.3. If, in (2.4), $G \equiv 0, \tau = 0$, $\lambda_1 > 0$ and $\lambda_2 = 0$, Theorem 2.1 (*under the metric* $\mathrm{d}_{\infty,1}$) becomes [22, Theorem 2.1], where the constant C in [22, Theorem 2.1] is independent of time $T = N$. While for the functional case, the constant $C > 0$ in Theorem 2.1 is dependent on time $T = N$ even for $\lambda_1 > 0$ (in (2.4)), which demonstrate the differences between SDEs without memory and functional SDEs. Nevertheless *under the metric* $\mathrm{d}_{\infty,2}$ Theorem 2.2 shows that the constant $C > 0$ such that TCI holds can be chosen independent of T. Moreover, following the argument of Theorem 2.1, our main result can also be generalized to the case of neutral functional with infinity delay, which will be reported in the future paper.

Remark 2.4. For time-inhomogenous diffusions, Üstünel [18, Proposition 1] and Pal [14, Theorem 5] verify that $\mathbb{P}_\xi \in T_2(C)$ with respect to $\mathrm{d}_{\infty,2}$ for the dissipative case and infinite time horizon case respectively. While in this section we discuss the Talagrand's T_2-transportation inequality with respect to two uniform metrics $\mathrm{d}_{\infty,1}$ and $\mathrm{d}_{\infty,2}$ for a class of *neutral functional* SDEs, and in particular some techniques have been developed to deal with the difficulties caused by the neutral term and the time-lag.

3 TCI for Neutral Functional SPDEs

In this section we proceed to discuss the TCIs for the laws of a class of neutral functional SPDEs in infinite-dimensional setting. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a real separable Hilbert space, and $(W(t))_{t\geq 0}$ a cylindrical Wiener process on H with respect to a filtered complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\mathscr{L}(H)$ and $\mathscr{L}_{HS}(H)$ be the spaces of all linear bounded operators and Hilbert-Schmidt operators on H respectively. Denote by $\|\cdot\|$ and $\|\cdot\|_{HS}$ the operator norm and the Hilbert-Schmidt norm respectively. Fix $\tau > 0$ and let $\mathscr{C} := \mathcal{C}([-\tau, 0]; H)$, the space of continuous functions $f : [-\tau, 0] \mapsto H$, equipped with a uniform norm $\|f\|_\infty := \sup_{-\tau\leq\theta\leq 0} \|f(\theta)\|_H$.

For $T > 0$ consider semi-linear neutral functional SPDE on H

eq19 (3.1)
$$\begin{cases} \mathrm{d}[X(t) + G(X_t)] = [AX(t) + b(X_t)]\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t), \quad t \in [0, T], \\ X_0 = \xi \in \mathscr{C}. \end{cases}$$

We assume that

(A1) $(A, \mathscr{D}(A))$ is a linear operator on H generating an analytic C_0-semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ such that $\|e^{tA}\| \leq Me^{\nu t}$ for some $M, \nu > 0$.

(A2) $b : \mathscr{C} \to H$ and there is $\rho_1 > 0$ such that $\|b(\xi) - b(\eta)\|_H \leq \rho_1\|\xi(\theta) - \eta(\theta)\|_\infty, \xi, \eta \in \mathscr{C}$.

(A3) $\sigma : H \to \mathscr{L}(H)$ such that $e^{tA}\sigma(0) \in \mathscr{L}_{HS}(H)$ for any $t \in [0, T]$ and $\int_0^T s^{-2\beta}\|e^{sA}\sigma(0)\|_{HS}^2\mathrm{d}s < \infty$ for some $\beta \in (0, \frac{1}{2})$. Moreover assume that there exists $\rho_2 > 0$ such that $\|\sigma(\xi) - \sigma(\eta)\|_{HS} \leq \rho_2\|\xi(\theta) - \eta(\theta)\|_\infty, \xi, \eta \in \mathscr{C}$.

(A4) $G : \mathscr{C} \to H$ such that $G(0) = 0$ and there exist $\alpha \in [0, 1]$ and $\rho_3 > 0$ such that $\|(-A)^\alpha G(\xi) - (-A)^\alpha G(\eta)\|_H \leq \rho_3\|\xi(\theta) - \eta(\theta)\|_\infty$, where $(-A)^\alpha$ is the fractional power of $-A$.

(A5) $\rho_3\Big(\|(-A)^{-\alpha}\| + M_{1-\alpha}\int_0^T \frac{e^{\nu t}}{t^{1-\alpha}}\mathrm{d}t\Big) < 1$, where $(-A)^{-\alpha}$ is the inverse of $-A$ and $M_{1-\alpha}$ is the positive constant in Lemma 3.1 below.

Remark 3.1. We remark from (*A*3) that $\sigma(\xi)-\sigma(\eta)\in\mathscr{L}_{HS}(H)$ while σ need not be Hilbert-Schmidt.

Note that $\int_0^T s^{-2\beta}\|e^{sA}\sigma(0)\|_{HS}^2\mathrm{d}s<\infty$ remains true by replacing β with a smaller positive number. So, we may take in (A3) $\beta\in(0,\frac{1}{p})$ for $p>2$. Denote by $\mathscr{H}_p$ the Banach space of all H-valued continuous adapted processes Y defined on the time interval $[-\tau,T]$ such that $X(t)=\xi(t), t\in[-\tau,0]$, and

$$\|X\|_p := \Big(\mathbb{E}\sup_{t\in[-\tau,T]}\|X(t)\|_H^p\Big)^{\frac{1}{p}}<\infty.$$

By the classical Banach-fixed-point-theorem approach, Eq. (3.1) admits a unique mild solution $\{X(t,\xi)\}_{t\in[0,T]}$. That is, for any $\xi\in\mathscr{C}$ there exists a unique H-valued adapted process $\{X(t,\xi)\}_{t\in[0,T]}$, which is continuous in $L^2(\Omega,\mathbb{P})$, such that

$$\begin{aligned}X(t)&=e^{tA}[\xi(0)+G(\xi)]-G(X_t)-\int_0^t Ae^{(t-s)A}G(X_s)\mathrm{d}s\\&\quad+\int_0^t e^{(t-s)A}b(X_s)\mathrm{d}s+\int_0^t e^{(t-s)A}\sigma(X_s)\mathrm{d}W(s).\end{aligned}$$

Remark 3.2. Since σ need not be Hilbert-Schmidt, the Burkhold-Davis-Gundy inequality [2, Proposition, p196] may not hold for $p=2$. Therefore the mild solution is shown by the fixed point theorem on Banach space $\mathscr{H}_p, p>2$.

The following lemma [13, Theorem 6.13] is vital to deal with the neutral term G.

Lemma 3 **Lemma 3.1.** Under (*A*1), for any $\beta\in(0,1]$ and $x\in\mathcal{D}((-A)^\beta)$

$$e^{tA}(-A)^\beta x=(-A)^\beta e^{tA}x$$

and there exists $M_\beta>0$ such that for any $t>0$

$$\|\,(-A)^\beta e^{tA}\,\|\le M_\beta t^{-\beta}e^{\nu t}.$$

Theorem 3 **Theorem 3.2.** Let (*A*1) – (*A*5) hold and $\mathbb{P}_\xi$ be the law of $X(\cdot,\xi)$, solution process of Eq. (3.1). Assume further that σ is bounded by $\tilde{\sigma}:=\sup_{\xi\in\mathscr{C}}\|\sigma(\xi)\|$. Then $\mathbb{P}_\xi\in T_2(C)$ for some $C>0$ on the metric space $\bar{\mathcal{X}}:=\mathcal{C}([0,T];H)$ with respect to the metric

$$\mathrm{d}_\infty(\gamma_1,\gamma_2):=\sup_{0\le t\le T}\|\gamma_1(t)-\gamma_2(t)\|_H,\quad \gamma_1,\gamma_2\in\bar{\mathcal{X}}.$$

Proof. We should point out that the former parts of the argument is similar to that of Theorem 2.1, while due to the unboundedness of infinitesimal generator A and the appearance of neutral term G, the Itô formula is not unavailable although $\sigma(\xi)-\sigma(\eta)\in\mathscr{L}_{HS}(H),\xi,\eta\in\mathscr{C}$.

In what follows we will use the theory of the semigroup. Let $\mathbb{P}_\xi$ be the law of $X(\cdot,\xi)$ on $\mathcal{X}$ and $\tilde{\mathbb{Q}}$ defined by (2.6). By the martingale representation theorem [2, Theorem 8.2, p220] we can also deduce that there exists a predictable process $h \in H$ with $\int_0^T \|h(s)\|_H^2 \mathrm{d}s < \infty$, $\mathbb{P}$-a.s., such that

$$\mathbf{H}(\tilde{\mathbb{Q}}|\mathbb{P}) = \mathbf{H}(\mathbb{Q}|\mathbb{P}_\xi) = \frac{1}{2}\mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^T \|h(t)\|_H^2 \mathrm{d}t.$$

Due to the Girsanov theorem

$$\tilde{W}(t) := W(t) - \int_0^t h(s)\mathrm{d}s, \quad t \in [0,T],$$

is a Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{Q}})$. Then, under the measure $\tilde{\mathbb{Q}}$, the process $\{X(t,\xi)\}_{t\in[0,T]}$ satisfies

eq20
$$\begin{cases} \mathrm{d}[X(t) + G(X_t)] = [AX(t) + b(X_t) + \sigma(X_t)h(t)]\mathrm{d}t + \sigma(X_t)\mathrm{d}\tilde{W}(t), \\ X_0 = \xi. \end{cases} \tag{3.2}$$

Let $\{Y(t,\xi)\}_{t\in[0,T]}$ be the solution of the following equation

eq21
$$\begin{cases} \mathrm{d}[Y(t) + G(Y_t)] = [AY(t) + b(Y_t)]\mathrm{d}t + \sigma(Y_t)\mathrm{d}\tilde{W}(t), \\ Y_0 = \xi. \end{cases} \tag{3.3}$$

By the uniqueness, under $\tilde{\mathbb{Q}}$ the law of $Y(\cdot)$ is $\mathbb{P}_\xi$. Thus (X,Y) under $\tilde{\mathbb{Q}}$ is a coupling of $(\mathbb{Q}, \mathbb{P}_\xi)$, and

$$(W_{2,\mathrm{d}}(\mathbb{Q}, \mathbb{P}_\xi))^2 \leq \mathbb{E}^{\tilde{\mathbb{Q}}}(\mathrm{d}_\infty(X,Y)).$$

Thus we only need to show

eq25
$$\mathbb{E}^{\tilde{\mathbb{Q}}}\Big(\sup_{0\leq t\leq T} \|X(t) - Y(t)\|_H^2\Big) \leq C\mathbb{E}^{\tilde{\mathbb{Q}}} \int_0^T \|h(t)\|_H^2 \mathrm{d}t. \tag{3.4}$$

Note from (3.2) and (3.3), together with the inequality (2.12), that for any $\epsilon \in (0,1)$

eq26
$$\begin{aligned} \|X(t) - Y(t)\|_H^2 \leq\ & \frac{1}{\epsilon}\Big\{\|G(Y_t) - G(X_t)\|_H + \Big\|\int_0^t Ae^{(t-s)A}[G(Y_s) - G(X_s)]\mathrm{d}s\Big\|_H\Big\}^2 \\ & + \frac{3}{1-\epsilon}\Big\{\Big\|\int_0^t e^{(t-s)A}[b(X_s) - b(Y_s)]\mathrm{d}s\Big\|_H^2 \\ & + \Big\|\int_0^t e^{(t-s)A}\sigma(X_s)h(s)\mathrm{d}s\Big\|_H^2\Big\} \\ & + \frac{3}{1-\epsilon}\Big\|\int_0^t e^{(t-s)A}[\sigma(X_s) - \sigma(Y_s)]\mathrm{d}\tilde{W}(s)\Big\|_H^2 \\ :=\ & I_1(t) + I_2(t) + I_3(t), \quad t \in [0,T], \end{aligned} \tag{3.5}$$

where we have also used the fundamental inequality $(a+b+c)^3 \leq 3(a^2+b^2+c^2)$ for $a, b, c \in \mathbb{R}$. Note that $(-A)^{-\alpha}$ is bounded by [13, Lemma 6.3, p71], and, in the light of [13, Theorem 6.8, p72],

$$(-A)^{\alpha+\beta}x = (-A)^{\alpha} \cdot (-A)^{\beta}x$$

for $x \in \mathcal{D}((-A)^{\gamma})$, the domain of $(-A)^{\gamma}$, with $\gamma := \max\{\alpha, \beta, \alpha+\beta\}$, $\alpha, \beta \in \mathbb{R}$. By (A4), it follows from Lemma 3.1 that

$$\begin{aligned}
\sup_{0\leq t\leq T} I_1(t) &= \frac{1}{\epsilon} \sup_{0\leq t\leq T} \Big\{ \|(-A)^{-\alpha}((-A)^{\alpha}G(Y_t) - (-A)^{\alpha}G(X_t))\|_H \\
&\quad + \Big\| \int_0^t (-A)e^{(t-s)A}(-A)^{-\alpha}[(-A)^{\alpha}G(Y_s) - (-A)^{\alpha}G(X_s)]\mathrm{d}s \Big\|_H \Big\}^2 \\
&= \frac{1}{\epsilon} \sup_{0\leq t\leq T} \Big\{ \|(-A)^{-\alpha}((-A)^{\alpha}G(Y_t) - (-A)^{\alpha}G(X_t))\|_H \\
&\quad + \Big\| \int_0^t A^{1-\alpha}e^{(t-s)A}[(-A)^{\alpha}G(Y_s) - (-A)^{\alpha}G(X_s)]\mathrm{d}s \Big\|_H \Big\}^2 \\
&\leq \frac{1}{\epsilon} \Big\{ \rho_3 \Big(\|(-A)^{-\alpha}\| + \int_0^T \|A^{1-\alpha}e^{tA}\|\mathrm{d}t \Big) \sup_{0\leq t\leq T} \|X(t) - Y(t)\|_H \Big\}^2 \\
&\leq \frac{1}{\epsilon} \Big\{ \rho_3 \Big(\|(-A)^{-\alpha}\| + M_{1-\alpha} \int_0^T \frac{e^{\nu t}}{t^{1-\alpha}}\mathrm{d}t \Big) \sup_{0\leq t\leq T} \|X(t) - Y(t)\|_H \Big\}^2,
\end{aligned}$$

where $X(t) = Y(t), t \in [-\tau, 0]$. Taking $\epsilon = \rho_2\Big(\|(-A)^{-\alpha}\| + M_{1-\alpha}\int_0^T \frac{e^{\nu t}}{t^{1-\alpha}}\mathrm{d}t\Big)$ we obtain from (A5) that

$$\sup_{0\leq t\leq T} I_1(t) \leq \epsilon \sup_{0\leq t\leq T} \|X(t) - Y(t)\|_H^2.$$

Thus due to (3.5)

eq22

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} \|X(t) - Y(t)\|_H^2\Big) \leq \frac{1}{1-\epsilon}\mathbb{E}\Big(\sup_{0\leq t\leq T} I_2(t)\Big) + \frac{1}{1-\epsilon}\mathbb{E}\Big(\sup_{0\leq t\leq T} I_3(t)\Big). \tag{3.6}$$

Next, by the Hölder inequality, (A2) and the boundedness of σ, we have

eq23

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\leq t\leq T} I_2(t)\Big) \leq \frac{3T}{1-\epsilon}\Big\{ \rho_1^2\tilde{M}^2 \int_0^T \mathbb{E}\Big(\sup_{0\leq s\leq t} \|X(s) - Y(s)\|_H^2\Big)\mathrm{d}t \\
+ \tilde{M}^2\tilde{\sigma}^2 \int_0^T \|h(s)\|_H^2 \mathrm{d}s \Big\},
\end{aligned} \tag{3.7}$$

where $\tilde{M} := M\sup_{t\in[0,T]}\|e^{tA}\|$. Furthermore by the Burkhold-Davis-Gundy inequality and (A3) there exists $C > 0$ such that

eq24

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} I_3(t)\Big) \leq \frac{3C\rho_2^2}{1-\epsilon}\int_0^T \mathbb{E}\Big(\sup_{0\leq s\leq t}\|X(s) - Y(s)\|_H^2\Big)\mathrm{d}t. \tag{3.8}$$

Then (3.4) follows by substituting (3.7) and (3.8) into (3.6) and applying the Gronwall inequality, and the proof is therefore complete. □

To demonstrate the applications of Theorem 3.2, we give an illustrative example motivated by [7, Example 4.1].

Example 3.3. Let $H := L^2([0,\pi])$, $A := \frac{\partial^2}{\partial x^2}$ with the domain $\mathcal{D} := H^2(0,\pi) \cap H_0^1(0,\pi)$, and $\{W(t,x), t \in [0,T], x \in [0,\pi]\}$ be a Brownian sheet defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., it is a centered Gaussian random field with the covariance $\mathbb{E}(W(t,x)W(s,y)) = (t \wedge s)(x \wedge y)$ for $s,t \in [0,T]$ and $x,y \in [0,\pi]$. Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be Lipschitzian, i.e., there exists $L > 0$ such that $|\phi(x) - \phi(y)| \leq L|x-y|, x,y \in \mathbb{R}$. Assume further that $\varphi : [-\tau, 0] \times [0,\pi] \times [0,\pi] \mapsto \mathbb{R}$ is measurable such that $\varphi(\cdot,\cdot,0) = \varphi(\cdot,\cdot,\pi) = 0$ and

eq04
$$N := \int_{-\tau}^0 \int_0^\pi \int_0^\pi \Big(\frac{\partial}{\partial x}\varphi(\theta,\xi,x)\Big)^2 \mathrm{d}\xi \mathrm{d}x \mathrm{d}\theta < \infty. \tag{3.9}$$

Consider neutral functional SPDE

eq01
$$\begin{aligned} &\frac{\partial}{\partial t}\Big[u(t,x) + \int_0^\pi \int_{-\tau}^0 \varphi(\theta,\xi,x)u(t+\theta,\xi)\mathrm{d}\theta\mathrm{d}\xi\Big] \\ &\quad = \Big\{\frac{\partial^2}{\partial x^2}u(t,x) + \phi\Big(\int_{-\tau}^0 u(t+\theta,x)\mathrm{d}\theta\Big)\Big\} + \frac{\partial^2 W}{\partial t \partial x}(t,x) \end{aligned} \tag{3.10}$$

with the Dirichlet boundary condition

$$X(t,0) = X(t,\pi) = 0, \quad t \in [0,T],$$

and the initial condition

$$X(\theta,x) = \psi(\theta,x), \quad \theta \in [-\tau,0], \quad x \in [0,\pi].$$

Recall that $e_n(x) := (2/\pi)^{1/2} \sin nx, n \in \mathbb{N}, x \in [0,\pi]$, is a complete orthonormal system of H, and that the eigenvector of A with eigenvalue $-n^2$, i.e., $Ae_n = -n^2 e_n$. Then we have

$$W(t)(x) := W(t,x) = \sum_{n=1}^\infty e_n(x) \int_0^t \int_0^\pi e_n(y) W(\mathrm{d}s, \mathrm{d}y),$$

which is a cylindrical Wiener process on H. For $t \in [0,T]$ and $x \in [0,\pi]$, let

$$X(t)(x) := u(t,x), \quad G(X_t)(x) := \int_0^\pi \int_{-\tau}^0 \varphi(\theta,\xi,x)X(t+\theta,\xi)\mathrm{d}\theta\mathrm{d}\xi$$

and

$$b(X_t)(x) := \phi\Big(\int_{-\tau}^0 X(t+\theta,x)\mathrm{d}\theta\Big).$$

Then Eq. (3.10) can be rewritten in the form (3.1). Observe that A generates a strongly continuous semigroup $\{e^{tA}\}_{t\in[0,T]}$, which is compact, analytic and self-adjoint, and

eq02
$$e^{tA}\xi = \sum_{n=1}^\infty e^{-n^2 t}\langle \xi, e_n\rangle e_n, \quad \xi \in H. \tag{3.11}$$

This gives that $\|e^{tA}\| \le e^{-t}$ and (A1) holds with $M = 1$ and $\nu = -1$. Moreover, due to ϕ is Lipschitzian we get from the Hölder inequality that

$$\begin{aligned}\|b(\xi) - b(\eta)\|_H^2 &\le L^2\tau \int_{-\tau}^0 \int_0^\pi (\xi(t+\theta, x) - \eta(t+\theta, x))^2 \mathrm{d}x\mathrm{d}\theta \\ &\le L^2\tau^2 \|\xi - \eta\|_H^2, \quad \xi, \eta \in H.\end{aligned}$$

Hence (A2) holds with $\rho_1 = L\tau$. By the definition of Hilbert-Schimdt, together with (3.11), it follows that

$$\int_0^T t^{-2\beta} \|e^{tA}\|_{HS}^2 \mathrm{d}s = \int_0^T t^{-2\beta} \sum_{n=1}^\infty \|e^{tA} e_n\|_H^2 \mathrm{d}s = \int_0^T t^{-2\beta} \sum_{n=1}^\infty e^{-2n^2 t} \mathrm{d}s.$$

Thus $(A3)$ holds for any $\beta \in (0, \frac{1}{4})$ and $\rho_2 = 0$. Furthermore note that

eq03
$$(-A)^{-\frac{1}{2}}\xi = \sum_{n=1}^\infty \frac{1}{n} \langle \xi, e_n \rangle e_n, \quad \xi \in H, \quad (-A)^{\frac{1}{2}}\xi = \sum_{n=1}^\infty n \langle \xi, e_n \rangle e_n, \quad \xi \in \mathcal{D}((-A)^{\frac{1}{2}}) \tag{3.12}$$

which in particular yields that $\|(-A)^{-\frac{1}{2}}\| = 1$. As a result, recalling that $\varphi(\cdot, \cdot, 0) = \varphi(\cdot, \cdot, \pi) = 0$, we derive from (3.9), (3.12) and the Hölder inequality that

$$\begin{aligned}\|(-A)^{\frac{1}{2}}(G(X_t) - G(Y_t))\|_H^2 &= \Big\| \sum_{n=1}^\infty n \langle G(X_t) - G(Y_t), e_n \rangle_H e_n \Big\|_H^2 \\ &= \sum_{n=1}^\infty \Big(n \int_0^\pi (G(X_t)(x) - G(Y_t)(x)) e_n(x) \mathrm{d}x \Big)^2 \\ &= \sum_{n=1}^\infty \Big(n \int_0^\pi \int_0^\pi \int_{-\tau}^0 \varphi(\theta, \xi, x) Z(t+\theta, \xi) \mathrm{d}\theta \mathrm{d}\xi e_n(x) \mathrm{d}x \Big)^2 \\ &= \sum_{n=1}^\infty \Big(\int_0^\pi \int_0^\pi \int_{-\tau}^0 \frac{\partial}{\partial x} \varphi(\theta, \xi, x) Z(t+\theta, \xi) \mathrm{d}\theta \mathrm{d}\xi \tilde{e}_n \mathrm{d}x \Big)^2 \\ &= \sum_{n=1}^\infty \Big\langle \int_0^\pi \int_{-\tau}^0 \frac{\partial}{\partial x} \varphi(\theta, \xi, \cdot) Z(t+\theta, \xi) \mathrm{d}\theta \mathrm{d}\xi, \tilde{e}_n \Big\rangle_H^2 \\ &= \Big\| \int_0^\pi \int_{-\tau}^0 \frac{\partial}{\partial x} \varphi(\theta, \xi, \cdot) Z(t+\theta, \xi) \mathrm{d}\theta \mathrm{d}\xi \Big\|_H^2 \\ &= \int_0^\pi \Big(\int_0^\pi \int_{-\tau}^0 \frac{\partial}{\partial x} \varphi(\theta, \xi, x) Z(t+\theta, \xi) \mathrm{d}\theta \mathrm{d}\xi \Big)^2 \mathrm{d}x \\ &\le \int_0^\pi \Big\{ \int_0^\pi \int_{-\tau}^0 \Big(\frac{\partial}{\partial x} \varphi(\theta, \xi, x) \Big)^2 \mathrm{d}\theta \mathrm{d}\xi \\ &\qquad \times \int_0^\pi \int_{-\tau}^0 Z^2(t+\theta, \xi) \mathrm{d}\theta \mathrm{d}\xi \Big\} \mathrm{d}x \\ &\le N\tau \|X_t - Y_t\|_\infty^2,\end{aligned}$$

where $Z(t) := X(t) - Y(t)$ and $\tilde{e}_n := \sqrt{\frac{2}{\pi}} \cos nx$, which is also a complete orthonormal system of H. Consequently the law of Eq. (3.10) $\mathbb{P}_\psi \in T_2(C)$ for some $C > 0$ under the metric d_∞ whenever $N\tau$ is sufficiently small.

Remark 3.3. In this paper we discuss TCIs for neutral functional SDEs and SPDEs driven by Brownian motion, where the Girsanov transformation plays an important role as we have explained. As pointed in [21, Remark 2.3], this approach is, however, unavailable for the jump-process cases. In the future, we shall establish by the Malliavin calculus method on the Poisson space the W_1H transportation inequalities for the distributions of the *segment processes* for a class of neutral functional SDEs with jumps.

Remark 3.4. There are many interesting applications of the TCIs, e.g., in Tsirel'son-type inequality and Hoeffding-type inequality, see [3, 22, 23], and in concentration of empirical measure [10, 21].

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